# Higgs decay into two photons, dispersion relations and trace anomaly

Jiří Hořejší[1] and Miroslav Stöhr[2]

*Nuclear Centre, Faculty of Mathematics and Physics, Charles University*

*V Holešovičkách 2, Prague 8, Czech Republic*

## Abstract

We examine the contribution of $W$ boson loops to the amplitude of the process $H \longrightarrow \gamma\gamma$ within the dispersion relation approach, taking up an issue raised in this context recently. We show that the non-vanishing limit of the relevant formfactor for $m_W \to 0$ is due to a finite subtraction induced by the value of the corresponding trace anomaly. The argument can be turned around and one thus arrives at a dispersive ("infrared") derivation of the trace anomaly.

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[1]e-mail: `Jiri.Horejsi@mff.cuni.cz`

[2]e-mail: `Miroslav.Stohr@mff.cuni.cz`

The decay process $H \longrightarrow \gamma\gamma$ occupies an important niche in the physics of Standard Model (SM), and it has been treated in the literature in considerable detail (see e.g.[1] – [6] and further references contained therein). In a recent paper, Körner et al. [7] considered the case of large Higgs mass and calculated some two-loop contributions to this process, in an approximation relying on the Goldstone-boson (GB) "equivalence theorem" for closed $W$ loops (which presumably should work in the heavy Higgs limit – cf. e.g. [8]), and using the technique of dispersion relations. Reproducing first the relevant one-loop results, the authors [7] pointed out some peculiar features of such a dispersive calculation, which remind one of the "infrared" aspects of the axial anomaly revealed through the imaginary part of the famous $VVA$ triangle graph [9], [10]. To put it in explicit terms, let us first introduce some notation. Let a (dimensionless) formfactor $F_W = F_W \left( m_W^2 / m_H^2 \right)$ describing the contribution of $W$ boson loops be defined through

$$\mathcal{M} \left( H \longrightarrow \gamma\gamma \right) = \frac{\alpha}{2\pi} \frac{1}{v} F_W \left( k.p \, g^{\mu\nu} - p^\mu k^\nu \right) \varepsilon_\mu^* \left( k \right) \varepsilon_\nu^* \left( p \right) \tag{1}$$

where $\alpha$ is the fine structure constant, $v = \left( G_F \sqrt{2} \right)^{-1/2}$ is the electroweak mass scale, and other symbols have an obvious meaning. In what follows we denote the $W$ mass simply as $m$ for brevity. The observation made in [7] consists in the following: If the $F_W$ is calculated within the "GB approximation" by means of unsubtracted dispersion relation, it does not vanish for $m_H \to \infty$ (or, equivalently, for $m \to 0$), while the corresponding imaginary part vanishes in the naive limit $m \to 0$. Thus, in the approach [7] one encounters another example of a situation where the suppression factor proportional to a mass is compensated after a pertinent integration; this is indeed closely analogous, at least technically, to the dispersive treatment of the axial anomaly [9], [10] (in particular, a relevant imaginary part becomes proportional to a $\delta$-function). The limit recovered in [7] within the GB approximation reads

$$\lim_{m \to 0} F_W \left( m^2 / m_H^2 \right) = 2 \tag{2}$$

which coincides with the value given by the exact one-loop calculation [2]. (Need-

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less to say, when considering the limit $m \to 0$, the $v$ is kept fixed.)

In this context, one may remember that the SM Higgs boson is coupled to the other fields through the trace of the "improved" energy-momentum tensor [11], i.e. essentially through the corresponding mass terms (this has been noticed already in the classic paper [1]). At quantum level, the trace of the energy-momentum tensor is known to be anomalous [12], [13] (i.e. it does not vanish in the massless limit when quantum corrections are taken into account) so one may wonder if the non-zero value of the limit (2) could be related to the well - known trace anomaly.

In the present note an attempt is made to elucidate this issue. We examine the dispersion relation (DR) approach to the relevant Feynman graphs and show that the limiting value (2) is determined by a "sum rule" for the imaginary part of the $W$ boson loops together with the value of the corresponding contribution to the trace anomaly. Later on we will also comment on the Goldstone boson approximation used in [7]. The order of the argument may then be reversed to arrive at a dispersive ("infrared") derivation of (the vector boson contribution to) the trace anomaly. Throughout our discussion we restrict ourselves to one-loop diagrams.

Let us start with an explicit expression for the imaginary part of the formfactor $F_W$ (defined by eq.(1)), corresponding to $W$ loops in the unitary gauge. This can be calculated by means of the Cutkosky rules which give the discontinuity associated with the standard cut $(4m^2, \infty)$ w.r.t. kinematical variable $t = (k + p)^2$ (see fig. 1). One gets

$$\text{Im}\, F_W\left(t; m^2\right) = \frac{3\pi}{2} \frac{4m^2}{t} \left(2 - \frac{4m^2}{t}\right) \ln \frac{1 + r}{1 - r} \tag{3}$$

for $t > 4m^2$, where we have denoted $r = (1 - 4m^2/t)^{1/2}$. Of course, the expression (3) coincides with that contained in the widely quoted result for the $F_W$ (cf. e.g. [4], [5]), obtained first in [2] by a direct Feynman graph calculation. We refer here to the alternative calculation via Cutkosky rules since one would like to avoid any explicit ultraviolet regularization when adopting the DR approach. With the
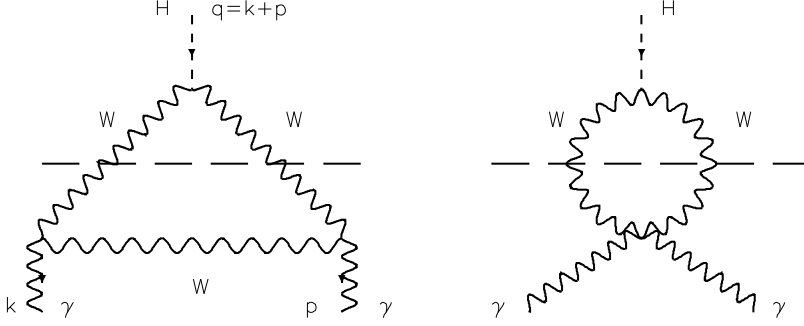
Figure 1: *Cut diagrams contributing to the imaginary part of the formfactor $F_W$ given in* Eq.(3)

imaginary part (3) at hand, one may try to evaluate the complete formfactor by means of a dispersion relation. Let us consider first the unsubtracted form

$$F_W^{(un)}\left(q^2; m^2\right) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im}\, F_W\left(t; m^2\right)}{t - q^2} dt \qquad (4)$$

for an arbitrary value of $q^2(= m_H^2)$. In view of (3) it is clear that the integral in the last expression is convergent (reflecting thus the fact that the total contribution of considered loops is ultraviolet finite in a renormalizable theory) but one may worry about possible finite subtractions in the considered DR if one wants to reproduce the result of the direct Feynman diagram calculation [2]. It turns out that such a subtraction is indeed necessary (in contrast with the scalar-loop method adopted in [7]), and its value may be deduced from the $W$ loop contribution to the trace anomaly. To see this, let us consider the limit of $F_W(q^2; m^2)$ for $q^2 \to 0$. One gets

$$F_W^{(un)}\left(0; m^2\right) = \frac{1}{\pi} \int_{4m^2}^{\infty} \text{Im}\, F_W\left(t; m^2\right) \frac{dt}{t} \qquad (5)$$

and the integral in the last expression is easily evaluated to give

$$\frac{1}{\pi} \int_{4m^2}^{\infty} \text{Im}\, F_W\left(t; m^2\right) \frac{dt}{t} = 5 \qquad (6)$$

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On the other hand, the value of $F_W(0; m^2)$ is fixed by a low-energy theorem (see [1] – [4], [6]), which leads to the conclusion that such a quantity is equal to the coefficient $b_W$ of the $\beta$-function associated with charge renormalization (due to the charged $W$ bosons in the present context) defined by

$$\beta_W(e) = -b_W \frac{1}{16\pi^2} e^3 \tag{7}$$

Let us emphasize that this is precisely the point where the trace anomaly enters the picture: In view of the structure of the SM Higgs couplings, the quantity $F_W(0; m^2)$ actually describes a two-photon matrix element of the trace of the energy-momentum tensor (cf. [1]) and this in turn is determined by the relevant $\beta$ - function (see [13]). At one-loop level one has $b_W = 7$ (see [2] and also the recent paper by Kniehl and Spira [6]), so we get a "boundary condition" for the DR representation of the formfactor $F_W$:

$$F_W\left(0; m^2\right) = 7 \tag{8}$$

Comparing now the last result with (5), (6), it is seen that the sum rule (6) does not saturate the value of the trace anomaly manifested in (8); it means that one has to include a finite subtraction in the DR representation of the $F_W$, namely

$$F_W\left(q^2; m^2\right) = F_W^{(un)}\left(q^2; m^2\right) + 2 \tag{9}$$

Using (3), it is not difficult to prove that

$$\lim_{m \to 0} F_W^{(un)}\left(q^2; m^2\right) = 0 \tag{10}$$

From (9) and (10) the limit (2) then follows immediately.

It is worth noticing that one has to make a finite subtraction in the dispersion relation for $W$ boson loops in the unitary gauge even though the relevant integral is convergent (for example, no such subtraction is needed for the contribution of a fermion loop to the same process). In a sense, the reasoning presented here bears some resemblance with the dispersive treatment of the axial anomaly: When adopting the usual tensor basis for the $VVA$ triangle graph [14], one also has

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to include finite subtraction in a convergent dispersion integral for the relevant formfactor if the axial anomaly is to be reproduced correctly (or, in other words, if the vector current is to be conserved).

Let us now return briefly to the observations made in [7]. In fact, the technique of "GB approximation" employed in [7] enables one to reformulate the preceding discussion concerning the trace anomaly in the following way. The contribution of $W$ boson loops may be calculated in a renormalizable ('t Hooft-Feynman) gauge. For $m_H^2 \gg m^2$, the contribution of the unphysical (Goldstone) scalar counterparts of the $W$ is supposed to be dominant and the leading contribution to the formfactor $F_W$ is evaluated by means of *unsubtracted* dispersion relation

$$F_W^{(un)}\left(q^2; m^2\right) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im}\, F_W^{(GB)}\left(t; m^2\right)}{t - q^2} dt \tag{11}$$

where, according to [7]

$$\text{Im}\, F_W^{(GB)}\left(t; m^2\right) = -\pi m_H^2 \frac{4m^2}{t^2} \ln \frac{1+r}{1-r} \tag{12}$$

with $r$ being defined as in (3). Note that the factor of $m_H^2$ in the last expression comes from the coupling constant for the interaction of the unphysical Goldstone scalars with physical Higgs boson. (To avoid confusion, let us also stress that the result (12) *does not* correspond to the expression for "physical" scalar loops usually quoted in the literature — see e.g. [4], [5] — precisely because of a different dependence of the relevant coupling constants on the masses involved.) In the limit $m \to 0$, the expression (12) becomes $-2\pi m_H^2 \delta(t)$ (in striking analogy with the calculation of the axial anomaly due to Dolgov and Zakharov [9]) and using (11) one then arrives immediately at the result (2). In such a way, the GB approximation provides a natural "infrared" explanation for the finite subtraction in (9): This may be understood as a contribution of unphysical Goldstone bosons (or, equivalently, of the longitudinal degrees of freedom associated with massive $W$ bosons). Now, with a simple interpretation of the subtraction constant in (9) at hand, one may invoke the sum rule (6) to recover eq.(8). Thus, from this point of view we have an alternative derivation of the "canonical" value of the

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$W$ boson contribution to the trace anomaly in terms of dispersion relations and "infrared" properties of the relevant one-loop graphs. Of course, such an analysis has one obvious flaw: It is not clear *a priori* that the unsubtracted DR in (11) is the correct form — in fact it is an "educated guess" which, strictly speaking, is only justified in confrontation with other results. Let us also remark that the Goldstone-boson equivalence theorem has been proved rigorously for the *external* vector bosons [15] (see also [16] for a recent reference), while its applicability for closed loops of massive vector bosons has only been tested on some specific examples (cf. e.g. [8]).

In concluding, let us summarize briefly the preceding considerations. Within the dispersion relation approach to $W$ boson loops (in the unitary gauge) the zero-mass limit (2) of the formfactor $F_W$ is determined by a finite subtraction, which is needed for reproducing the correct value of the trace anomaly fixed by a well-known low-energy theorem. In this way, the result (2) emerges as the difference between the anomaly value and a sum rule for the imaginary part of the $F_W$ (see (5)). On the other hand, using the approximation invoked in [7] one gets an "infrared" interpretation of the limit (2) as an effect of the (would-be) Goldstone bosons associated with the $W$. Thus, the previous reasoning can be turned around: Using the insight provided by the GB approximation [7] together with the sum rule (5), (6) one arrives at an alternative (dispersive) derivation of the $W$ boson contribution to the trace anomaly.

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